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Yang-Mills duals for semiclassical strings on $AdS_5 \times S_5$

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Abstract: We consider a semiclassical multiwrapped circular string pulsating on S_5, whose center of mass has angular momentum J on an S_3 subspace. Using the AdS/CFT correspondence we argue that the one-loop anomalous dimension of the dual operator is a simple rational function of J/L, where J is the R-charge and L is the bare dimension of the operator. We then reproduce this result directly from a super Yang-Mills computation, where we make use of the integrability of the one-loop system to set up an integral equation that we solve. We then verify the results of Frolov and Tseytlin for circular rotating strings with R-charge assignment (J', J', J). In this case we solve for an integral equation found in the $O(-1)$ matrix model when $J' < J$ and the $O(+1)$ matrix model if $J' > J$. The latter region starts at $J' = L/2$ and continues down, but an apparent critical point is reached at $J' = 4J$. We argue that the critical point is just an artifact of the Bethe ansatz and that the conserved charges of the underlying integrable model are analytic for all J' and that the results from the $O(-1)$ model continue onto the results of the $O(+1)$ model.

Keywords: AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

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1. Introduction

While the AdS/CFT conjecture [1]–[3] is generally accepted as fact, proving it is a highly nontrivial problem. However, there are many nontrivial tests that can be applied to the conjecture, and these tests may provide insight toward a formal proof. Furthermore, working out the consequences of the AdS/CFT duality in concrete examples uncovers a beautiful interplay between quantum fields and strings.

The most celebrated "stringy" test of the conjecture is the comparison of the string spectrum in the plane-wave limit [4, 5] to the anomalous dimension of single trace operators that differ from BPS protected operators by a finite amount [6]. More recently, a program has begun comparing the spectrum of semiclassical string configurations [7]–[15] with anomalous dimensions for a wider class of operators [16]–[22]. These operators also have large charges, but are not necesserily close to BPS operators.

One such comparison was made between a folded string in S_5, rotating in one plane and revolving in another [14, 15], and a single trace operator composed of two types of scalar fields [18]. The energy of the string corresponds to the scaling dimension of an operator on the gauge theory side, and this was found by mapping the problem to the Heisenberg spin chain, where the anomalous dimension can be computed as an eigenvalue of the spin hamiltonian [16]. The integrability of the Heisenberg model is a powerful tool, allowing one to reduce the problem to solving a series of Bethe equations [23, 24]. The semiclassical limit of the string corresponds to the thermodynamic limit in the spin system, which allows one to convert the Bethe equations into a relatively simple integral equation. This has a striking resemblance to saddle-point equations in certain large-N matrix models and is solved by similar techniques. The result is a parametric relationship between the anomalous dimension and the R-charges of the Super Yang-Mills (SYM) operator [18]. The same relationship between the energy and the angular momenta arises in the string computation [14, 15]. This result was generalized to folded strings that rotate not only in S_5, but also in the AdS_5 [21], again showing a beautiful agreement between string theory and SYM.

The string states on S_5 and the scalar operators in the SYM theory can be characterized by three R-charges which define the highest weight (J_1, J_2, J_3) of an SO(6) representation with Dynkin indices $[J_2+J_3, J_1-J_2, J_2-J_3]$. On the string side, the R-charges correspond to the angular momenta on S_5. The simplest operator in the SYM theory with the R-charge assignment (J_1, J_2, J_3) is $\mathrm{Tr}\,(X^{J_1}Y^{J_2}Z^{J_3})$, where X, Y and Z are the standard complex scalars of the $\mathcal{N}=4$ supermultiplet,

$$X = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad Y = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4) \qquad Z = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6)\,. \tag{1.1}$$

The R-charge does not depend on the ordering in the trace but operators with different ordering mix under renormalization. This mixing makes computation of one-loop anomalous dimensions a nontrivial problem. The efficient way to resolve the operator mixing is to map the problem to the hamiltonian of an integrable SO(6) spin chain [16], which is then solved using the Bethe ansatz. The bare dimension of a holomorphic operator, which does not contain $\overline{X}$, $\overline{Y}$ or $\overline{Z}$, is a sum of the R-charges. This property resembles BPS saturation, and indeed some of the string solitons of [12]–[15] are supersymmetric in the tensionless limit of string theory [25], the limit that is supposed to be dual to free SYM. We strongly believe this asymptotic supersymmetry is important but not really necessary for establishing the precise agreement between semiclassical string states and SYM operators. To support this point of view, we will give examples of string states that cannot satisfy a BPS bound by dimensional counting. We will construct AdS duals of these states in a way similar to the construction of duals for the asymptotically supersymmetric states. The corresponding SYM operators will contain both holomorphic and anti-holomorphic fields, therefore, their bare dimension will *exceed* their R-charge by an arbitrarily large amount.

Plenty of string solutions with all possible charge assignments are known [15]. Some of these solutions predict a simple analytic relation between the anomalous dimension and the R-charges or the bare dimension. We will concentrate on the duals of these string states.

For example, the folded string state of [14, 15] has an SYM dual with R-charge assignment $(J', 0, J)$. The string prediction was shown to match with the one-loop SYM result, but the relation between the anomalous dimension and the ratio of J'/J involved elliptic functions [14, 18, 21]. When $J' = J$, the folded string has the same R-charges as a circular string that wraps around two planes and is stationary in a third [12, 13]. In [18] the SYM dual for this string was found, where not only the anomalous dimension was shown to be the same, but the fluctutation spectrum was also shown to match. But it is also simple to compute the energy of a circular string when there *is* a center of mass motion in the third plane, and the R-charges are of the form (J', J', J). Expanding about small 't Hooft coupling $\lambda = g_{YM}^2 N$, one finds that the corresponding anomalous dimension has the particularly simple form

$$\gamma = \frac{m^2\lambda J'}{L^2}\,, \tag{1.2}$$

where $L = 2J' + J$ and m counts the string winding. Given this extraordinarily simple equation, it seems likely that such a result can be reproduced by a one-loop SYM computation. One of the goals of this paper is do precisely this.

We will also identify an even simpler duality. We consider a string that is pulsating between two extremal points on S_5, where the string also has center of mass motion along an S_3 subspace, giving it angular momentum J. This is a generalization of a case previously considered in [10]. Using simple first order perturbation theory, we will show that the prediction for the one-loop anomalous dimension is

$$\gamma = \frac{m^2\lambda(L^2 - J^2)}{4L^3}\,, \tag{1.3}$$

We then reproduce this equation for a single trace operator with R-charge assignment $(0, 0, J)$ and bare dimension L. It is not necessary that J be close to L.

From the SYM point of view, we will see that equations eq. (1.2) and eq. (1.3) are simple because the integral equations reduce to those found with $O(n)$ matrix models [26]–[32]. For these models, if n is parameterized as

$$n = 2\cos\left(\frac{\pi p}{q}\right) \tag{1.4}$$

with p and q positive integers having no common factor, then the resolvant of the eigenvalue density is the solution of a polynomial equation of order q. We will see that the Bethe equations for the dual of the pulsating string reduce to the integral equation of an $O(0)$ model with a critical point at $J = 0$. This case is particularly simple since $p/q = 1/2$.

For the duals of the circular string, we need to consider two regions. The first region has $J' < J$ and the Bethe equations reduce to an $O(-1)$ integral equation. The second region has $J' > 4J$, where after some work, the Bethe equations are reduced to an $O(+1)$ integral equation. The point $J' = 4J$ corresponds to a critical point in the $O(+1)$ model, but we believe that this is only an artifact of the Bethe ansatz and not a true critical point for the SYM operators. Indeed, one can analytically continue our results for $J' < J$ up to $J' = 4J$, where the $O(-1)$ model also has a critical point, and then continue beyond this point to $J = 0$. We find that not only is the anomalous dimension analytic across the critical point but so are all the conserved charges of the underlying integrable model, and that they match onto the $O(+1)$ results.

The paper is organized as follows. In section 2 we compute the energy spectrum for the string pulsating on S_5. In section 3 we consider the one-loop anomalous dimensions for R-charge assignments $(0, 0, J)$ and (J', J', J) with $J' < J$, where the bare dimension is $L > J$ in the first case and $L = 2J' + J$ in the second. In sections 4 and 5 we solve the integral equations for these cases and show that the anomalous dimensions are given by eq. (1.3) and eq. (1.2). In section 6 we consider the case with R-charge assignment (J', J', J) and $J' > J$, and show that the anomalous dimension is still given by eq. (1.2). In section 7 we present our conclusions.

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2. A string pulsating on S_5

In this section we generalize the results of [10] to include an R-charge. Let us consider a circular pulsating string expanding and contracting on S_5, which has a center of mass that is moving on an S_3 subspace. We will assume that the string is fixed on the spatial coordinates in AdS_5, so the relevant metric for us is

$$ds^2 = R^2(-dt^2 + \sin^2\theta\, d\psi^2 + \ d\theta^2 \ + \ \cos^2\theta\, d\Omega_3^2)\,, \tag{2.1}$$

where $d\Omega_3$ is the metric on the S_3 supspace and $R^2 = 2\pi\alpha'\sqrt{\lambda}$. We will assume that the string is stretched along ψ but not along any of the coordinates in S_3. If we identify t with τ and ψ with $m\sigma$ to allow for multiwrapping, the Nambu-Goto action then reduces to

$$S \ = - \ m\sqrt{\lambda}\int dt\ \sin\theta\ \sqrt{1-\dot{\theta}^2-\cos^2\theta g_{ij}\dot{\phi}^i\dot{\phi}^j}\,, \tag{2.2}$$

where g_{ij} is the metric on S_3 and ϕ^i refers to the coordinates on S_3. Hence, the canonical momentum are

$$\begin{aligned}\Pi_\theta \ &= \ \frac{m\sqrt{\lambda}\ \sin\theta\ \dot{\theta}}{\sqrt{1-\dot{\theta}^2-\cos^2\theta g_{ij}\dot{\phi}^i\dot{\phi}^j}}\,,\\ \Pi_i \ &= \ \frac{m\sqrt{\lambda}\ \sin\theta\ \cos^2\theta g_{ij}\dot{\phi}^j}{\sqrt{1-\dot{\theta}^2-\cos^2\theta g_{ij}\dot{\phi}^i\dot{\phi}^j}}\,.\end{aligned} \tag{2.3}$$

Solving for the derivatives in terms of the canonical momenta and substituting into the hamiltonian, we find

$$H = \sqrt{\Pi_\theta^2 + \frac{g^{ij}\Pi_i\Pi_j}{\cos^2\theta} + m^2\lambda\sin^2\theta}\,. \tag{2.4}$$

The square of H looks like the hamiltonian for a particle on S_5 with an angular dependent potential. Since we are interested in large quantum numbers, the potential may be considered a perturbation. We thus proceed by considering free wavefunctions on S_5 and then do first order perturbation theory to find the order λ correction. The total S_5 angular momentum quantum number will be denoted by L and the total angular momentum quantum number on S_3 is J. Since the potential depends only on θ, we may replace $g^{ij}\Pi_i\Pi_j$ with $J(J+2)$.

The wave-functions are solutions to the Schrödinger equation

$$L(L+4)\Psi(w) = -\frac{4}{w}\frac{d}{dw}w^2(1-w)\frac{d}{dw}\Psi(w) + \frac{J(J+2)}{w}\Psi(w)\,, \tag{2.5}$$

where $w = \cos^2\theta$. In order to simplify the discussion, we will assume that J and L are even and define $j = J/2$ and $\ell = L/2$. In this case, the normalized S_5 wave functions are given by

$$\Psi(w) = \frac{\sqrt{2(\ell+1)}}{(\ell-j)!}\frac{1}{w^{j+1}}\left(\frac{d}{dw}\right)^{\ell-j} w^{\ell+j}(1-w)^{\ell-j}\,. \tag{2.6}$$

Hence, the first order correction to E^2 is

$$\int_0^1 wdw\Psi(w)m^2\lambda(1-w)\Psi(w) = m^2\lambda\frac{2(\ell+1)^2-(j+1)^2-j^2}{(2\ell+1)(2\ell+3)}\,. \tag{2.7}$$

Thus, up to first order in λ and assuming L and J large, E^2 is given by

$$E^2 = L(L+4) + m^2\lambda\frac{L^2-J^2}{2L^2}\,. \tag{2.8}$$

Now the bare dimension is L, and so the anomalous dimension is given by

$$\gamma = \frac{m^2\lambda}{4L}\alpha(2-\alpha)\,, \tag{2.9}$$

where $\alpha = 1 - J/L$, which we have defined for later convenience. In the next section we will reproduce this result in a one-loop SYM computation.

Even though L is nominally a quantum number on S_5, it is not the R-charge. This is because the wave function chosen is for a rigid string, not a particle. Any contribution to the total angular momentum for a section of string moving along θ is cancelled by the section halfway around the string. Only the quantum number J on S_3 contributes to the R-charge since we are assuming that the string is not stretched along here.

3. Setting up the gauge theory computations

In this section, we derive integral equations for the gauge theory computations that will be solved for in the subsequent two sections. In order to do these computations, we capitalize on the fact that the one-loop anomalous dimension can be mapped to a hamiltonian of an integrable spin chain [16]. With this, we can write down a set of Bethe equations that can be solved in the limit that the number of sites in the chain is large. We will consider single trace operators $\mathcal{O}$ made up of scalar fields only. The operators are not required to be holomorphic, but can contain $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ scalar fields inside the trace. We will assume that the operators are highest weights of SO(6) representations and that the R-charges have the general form (J', J', J). In terms of SO(6) Dynkin indices, these representations are denoted by $[0, J-J', 2J']$ if $J' \leq J$ and $[J'-J, 0, J'+J]$ if $J' \geq J$. If $J' \neq 0$, then we will assume that the bare dimension L of $\mathcal{O}$ maximizes the BPS-like bound $L = J + 2J'$. If $J' = 0$ we will relax this condition.

In [16] it was argued that the anomalous dimension of $\mathcal{O}$ can be found by solving a series of Bethe equations for a set of Bethe roots. There are three types of Bethe roots, with each type associated with a simple root of the SO(6) Lie algebra. Assuming that there are L scalar fields in $\mathcal{O}$, the Bethe equations are given by

$$\begin{aligned}
\left(\frac{u_{1,i}+i/2}{u_{1,i}-i/2}\right)^L &= \prod_{j\neq i}^{n_1}\frac{u_{1,i}-u_{1,j}+i}{u_{1,i}-u_{1,j}-i}\prod_{j}^{n_2}\frac{u_{1,i}-u_{2,j}-i/2}{u_{1,i}-u_{2,j}+i/2}\prod_{j}^{n_3}\frac{u_{1,i}-u_{3,j}-i/2}{u_{1,i}-u_{3,j}+i/2}\\
1 &= \prod_{j\neq i}^{n_2}\frac{u_{2,i}-u_{2,j}+i}{u_{2,i}-u_{2,j}-i}\prod_{j}^{n_1}\frac{u_{2,i}-u_{1,j}-i/2}{u_{2,i}-u_{1,j}+i/2}\\
1 &= \prod_{j\neq i}^{n_3}\frac{u_{3,i}-u_{3,j}+i}{u_{3,i}-u_{3,j}-i}\prod_{j}^{n_1}\frac{u_{3,i}-u_{1,j}-i/2}{u_{3,i}-u_{1,j}+i/2}\,,
\end{aligned} \tag{3.1}$$

where n_1, n_2 and n_3 denote the number of Bethe roots associated with each simple root of SO(6). For this choice, the Dynkin indices of this representation are given by $[n_1-2n_2, L-2n_1+n_2+n_3, n_1-2n_3]$. The anomalous dimension is only directly related to the u_1 roots and is given by

$$\gamma = \frac{\lambda}{8\pi^2}\sum_i^{n_1}\frac{1}{(u_{1,i})^2+1/4}\,. \tag{3.2}$$

Given our restrictions on the R-charges, we will only consider three cases. These are (i) $n_2 = n_3 = n_1/2$ and so the representation is $[0, L-n_1, 0]$, with $J_1 = J$, $J_2 = J_3 = 0$ and $n_1 = L - J$. (ii) $n_2 = n_1/2$, $n_3 = 0$, and so the representation is $[0, L-n_1-n_2, 2n_2]$ with $J_1 = J$, $J_2 = J_3 = J'$, $n_2 = J'$ and $n_1 = L - J$. (iii) $n_1 = L/2 + n_2/2$, $n_3 = 0$ and so the representation is $[n_1 - 2n_2, 0, n_1]$ with $J_1 = J_2 = J'$, $J_3 = J$, $J' = n_1 - n_2$ and $J = n_2$.

In the first two cases we will be looking for the operator with the lowest anomalous dimension for a given set of R-charges and bare dimension. For this reason the distribution of the roots will be highly symmetric. In the third case, we will not have the lowest anomalous dimension for the given representation, but we will still have a symmetric distribution of roots. We shall see that the cases (ii) and (iii) are related by analytic continuation. In the course of the analytic continuation a level crossing should occur where another branch of semiclassical states becomes the global minimum of the anomalous dimension in the (J', J', J) sector. These semiclassical states are the dual of the folded string when $J \to 0$.

In the rest of this section we will consider cases (i) and (ii). Case (iii) will be discussed in a later section.

We proceed as in [18], where we assume the number of roots is of order L. We assume the roots are equally distributed about $u = 0$ and the distribution of u_1 roots is of the same form as in [18], with the roots separated into two symmetric curves that intersect the real axis. Taking logs of the equations in eq. (3.1), rescaling u by a factor of L and replacing sums by integrals, we are left with the following equations:

$$\begin{aligned}
\frac{1}{u} - 2\pi m &= \alpha \fint_{\mathcal{C}_+} du'\frac{\sigma(u')}{u-u'} + \alpha\int_{\mathcal{C}_+} du'\frac{\sigma(u')}{u+u'} - \beta\int_{\mathcal{C}_2} du'\frac{\rho_2(u')}{u-u'} - \beta'\int_{\mathcal{C}_3} du'\frac{\rho_3(u')}{u-u'}\\
0 &= 2\beta\fint_{\mathcal{C}_2} du'\frac{\rho_2(u')}{u-u'} - \frac{\alpha}{2}\int_{\mathcal{C}_+} du'\frac{\sigma(u')}{u-u'} - \frac{\alpha}{2}\int_{\mathcal{C}_+} du'\frac{\sigma(u')}{u+u'}\\
0 &= 2\beta'\fint_{\mathcal{C}_3} du'\frac{\rho_3(u')}{u-u'} - \frac{\alpha}{2}\int_{\mathcal{C}_+} du'\frac{\sigma(u')}{u-u'} - \frac{\alpha}{2}\int_{\mathcal{C}_+} du'\frac{\sigma(u')}{u+u'}\,,
\end{aligned} \tag{3.3}$$

where $\alpha = n_1/L$, $\beta = n_2/L$ and $\beta' = n_3/L$. $\mathcal{C}_+$ is the right contour for the u_1 roots, $\mathcal{C}_2$ is the contour for the u_2 roots and $\mathcal{C}_3$ is the contour for the u_3 roots. The left contour of the u_1 roots, $\mathcal{C}_-$, is assumed to be the mirror image of $\mathcal{C}_+$. The root densities are normalized to

$$\int_{\mathcal{C}_+}\sigma(u')du' = \int_{\mathcal{C}_2}\rho_2(u')du' = \int_{\mathcal{C}_3}\rho_3(u')du' = 1\,. \tag{3.4}$$

If we think of the Bethe roots as corresponding to the positions of three different types of particles, then the solutions to the integral equations in eq. (3.3) give their equilibrium positions. From these equations we see that particles of the same type repulse each other.

Also, the first type of particles are attracted to the other two types and are also attracted to a potential that has a minimum at $u = \pm(2\pi m)^{-1}$. The particles of the second and third type do not see the potential, nor do they interact directly with each other. Assuming the particles of the first type are arranged in two equal curves intersecting the real axis, then the particles of the second and third type must lie on the imaginary axis.

We can thus solve for $\rho_2(u)$ and $\rho_3(u)$ in terms of $\sigma(u)$ and substitute the result back into the first equation in eq. (3.3). Performing Hilbert transforms, we find that

$$\rho_2(iu) = -\frac{\alpha}{2\pi^2\beta}\sqrt{c^2-u^2}\fint_{-c}^{c}\frac{du'}{u-u'}\frac{u'}{\sqrt{c^2-u^2}}\int_{\mathcal{C}_+}du''\frac{\sigma(u'')}{(u')^2+(u'')^2} \tag{3.5}$$

and a similar equation for ρ_3. Inverting the order of integration and deforming the contour, we find

$$\rho_2(iu) = \frac{\alpha}{2\beta\pi}\int_{\mathcal{C}_+}du''\frac{\sigma(u'')u''}{u^2+(u'')^2}\frac{\sqrt{c^2-u^2}}{\sqrt{c^2+(u'')^2}}\,. \tag{3.6}$$

To determine c, we plug eq. (3.6) into eq. (3.4), where we find

$$\begin{aligned}\int_{-c}^{+c}du'\rho_2(iu') &= \frac{\alpha}{2\beta}\int_{\mathcal{C}_+}du'\sigma(u') - \frac{\alpha}{2\beta}\int_{\mathcal{C}_+}du'\frac{\sigma(u')}{\sqrt{c^2+u'^2}} \\ &= \frac{\alpha}{2\beta} - \frac{\alpha}{2\beta}\int_{\mathcal{C}_+}du'\frac{\sigma(u')}{\sqrt{c^2+u'^2}} = 1\,.\end{aligned} \tag{3.7}$$

If $\beta = \alpha/2$, then $c = \infty$. Assuming this "half-filling" condition, one can easily show that

$$u\int_{-\infty}^{+\infty}du'\frac{\rho_2(iu')}{u^2+(u')^2} = \int_{C_+}du'\frac{\sigma(u')}{u+u'}\,. \tag{3.8}$$

It is also clear that if the u_3 roots are half-filled, then $\rho_3(u)$ satisfies the same equation. Hence, if both sets of roots are half-filled then this is case (i) and the first integral equation in eq. (3.3) reduces to

$$\frac{2}{\alpha}\left(\frac{1}{u} - 2\pi m\right) = 2\fint_{\mathcal{C}_+}du'\frac{\sigma(u')}{u-u'}\,. \tag{3.9}$$

If the u_2 roots are half-filled and there are no u_3 roots then this is case (ii) and the first integral equation in eq. (3.3) reduces to

$$\frac{2}{\alpha}\left(\frac{1}{u} - 2\pi m\right) = 2\fint_{\mathcal{C}_+}du'\frac{\sigma(u')}{u-u'} + \int_{\mathcal{C}_+}du'\frac{\sigma(u')}{u+u'}\,. \tag{3.10}$$

The eigenvalue density for an $O(n)$ matrix model satisfies the integral equation [26, 27]

$$U'(u) = 2\fint_b^a du'\frac{\sigma(u')}{u-u'} - n\int_b^a du'\frac{\sigma(u')}{u+u'}\,. \tag{3.11}$$

If we compare eq. (3.11) to eq. (3.9) and eq. (3.10), we see that these are both of this form with

$$U'(u) = \frac{2}{\alpha}\left(\frac{1}{u} - 2\pi m\right)\,. \tag{3.12}$$

and $n = 0$ for case (i) and $n = -1$ for case (ii).

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4. The gauge dual of the pulsating string

We start with case (i) since this is simpler. This has only one R-charge J and a bare dimension $L > J$. We claim that this is the SYM dual to the pulsating string.

We first do a Hilbert transform on $\sigma(u)$ in eq. (3.9), giving

$$\sigma(u) = -\frac{\sqrt{(a-u)(u-b)}}{\pi\alpha u\sqrt{ab}}\,, \tag{4.1}$$

where a and b are the end points of the cut. These can be determined from eq. (3.4) which gives

$$\frac{a+b}{\sqrt{ab}} = 2(1-\alpha)\,, \tag{4.2}$$

and by explicitly plugging eq. (4.1) into eq. (3.9), which gives

$$⨍_b^a du' \frac{\sigma(u')}{u-u'} = \frac{1}{\alpha}\left(\frac{1}{u} - \frac{1}{\sqrt{ab}}\right). \tag{4.3}$$

Hence we have

$$\sqrt{ab} = \frac{1}{2\pi m} \qquad\qquad a+b = \frac{1}{\pi m}(1-\alpha)\,. \tag{4.4}$$

For negative α, we see that a and b are both real. When α is positive then $b = a^*$.

In order to find the anomalous dimension γ it is convenient to define the resolvent $W(u)$

$$W(u) = \int_b^a du' \frac{\sigma(u')}{u-u'}\,. \tag{4.5}$$

Using eq. (4.1) and eq. (4.4), we find that

$$W(u) = \frac{1}{u}\left(1 - \sqrt{(1-2\pi m u)^2 + 2\alpha(2\pi m u)}\right) - 2\pi m\,. \tag{4.6}$$

From eq. (3.2), eq. (4.5) and eq. (4.6) we see that γ is given by

$$\gamma = -\frac{\lambda}{8\pi^2 L}\alpha W'(0) = \frac{\lambda m^2}{4L}\alpha(2-\alpha)\,, \tag{4.7}$$

precisely matching the result from the previous section.

We can say a bit more about our result. First, there is a critical point at $\alpha = 1$. This is the point where the representation is the SO(6) singlet, so it is not surprising to find the critical behavior here. At this critical point, we find that $a = -b = i/2\pi m$. Hence at this value, the contour $\mathcal{C}_+$ is touching the imaginary axis and its mirror $\mathcal{C}_-$.

The simplicity of the root distribution also makes it easy to consider the higher conserved charges. The generator of higher charges is [24]

$$t(u) = \sum_n t_n u^n = i \log\left(\prod_k^{n_1} \frac{u - u_{1,k} + i/2}{u - u_{1,k} - i/2}\right), \tag{4.8}$$

Under the rescaling, this reduces to

$$t(u) = -\frac{\alpha}{2}\left(W(u) - W(-u)\right), \tag{4.9}$$

and the charges t_n are rescaled by a factor of L^{-n}. We have already seen that γ is related to the linear coefficient in $t(u)$. The next nontrivial charge is

$$t_3 = \frac{(2\pi m)^4}{8L^3}\alpha(2-\alpha)(5(1-\alpha)^2-1)\,. \tag{4.10}$$

What this corresponds to on the string side is not immediately clear.

Even though half the charges are zero, we can still glean some information from them. For example, the total momentum on the string must be zero because of level matching. In SYM, this corresponds to the cyclicity property of the trace. But we can still determine the left and right moving contribution to the momentum. This is just

$$-\frac{\alpha L}{2}W(0) = 2\pi m\left(\frac{\alpha L}{2}\right). \tag{4.11}$$

In other words, the left moving momentum is $2\pi m$ multiplied by half the number of impurities. This is the result in the BMN limit, so for this configuration, there is no correction even with a large number of impurities.

We can also go back and compute $\rho_2(iu) = \rho_3(iu)$, where we find

$$\rho_2(iu) = \frac{2m}{\alpha}\left(1+\frac{\sqrt{(a-iu)(b-iu)}}{2iu}-\frac{\sqrt{(a+iu)(b+iu)}}{2iu}\right). \tag{4.12}$$

In the limit $\alpha \to 1$, this reduces to

$$\begin{aligned}\rho_2(iu) &= 2m & -\frac{1}{2\pi m} \leq u \leq \frac{1}{2\pi m}\\ &= 2m\left(1-\sqrt{1-(2\pi m u)^{-2}}\right) & |u| > \frac{1}{2\pi m}\,.\end{aligned} \tag{4.13}$$

This is the same distribution of u_2 and u_3 roots found in [18] for an SO(6) singlet. In this case the u_1 roots were also on the imaginary axis with a density twice that of the other two roots. Comparing the $\alpha = 1$ result for $t(u)$ in eq. (4.9) to the corresponding generator for the imaginary root solution, one finds that they match. This suggests that imaginary root Bethe state is equivalent to the $\alpha = 1$ Bethe state here.

5. The SYM dual of the Frolov-Tseytlin string (I)

Let us now consider case (ii) with equation eq. (3.10). This is the SYM dual to the Frolov-Tseytlin string [12] when $J' < J$. In order to solve this, we use the results in [29].

The resolvent $W(u)$ is analytic everywhere except across the cut on $\mathcal{C}_+$. On the cut, it follows from eq. (3.10) that the resolvent satisfies the equation

$$W(u+i0)+W(u-i0)-W(-u) = U'(u)\,. \tag{5.1}$$

A solution to this equation is

$$W_r(u) = \frac{1}{3}\left(2U'(u)+U'(-u)\right) = \frac{2}{3\alpha\, u}-\frac{4\pi m}{\alpha}\,. \tag{5.2}$$

We will thus assume that

$$W(u) = W_r(u) + w(u)\,, \tag{5.3}$$

where $w(u)$ satisfies the homogeneous equation

$$w(u+i0) + w(u-i0) = w(-u)\,. \tag{5.4}$$

If we now consider the function $r(u)$,

$$r(u) \equiv w^2(u) - w(u)w(-u) + w^2(-u)\,, \tag{5.5}$$

then it is simple to show that $r(u)$ is an even function that is regular across the cut. Given the form of $W_r(u)$ in eq. (5.2), $r(u)$ approaches a constant at infinity and has a double pole at $u = 0$. The form is easily determined by recalling that $W(u)$ is regular at $u = 0$, and falls off as $1/u$ for large u. Thus, $w(u)$ must be chosen to cancel off the constant piece of $W_r(u)$ for large u and the pole at $u = 0$, as well as matching onto the correct asymptotic behavior for $W(u)$. This gives us

$$\begin{aligned} w(u) &\approx \frac{4\pi m}{\alpha} + \left(1 - \frac{2}{3\alpha}\right)\frac{1}{u} && u \to \infty \\ w(u) &\approx -\frac{2}{3\alpha u} && u \to 0\,, \end{aligned} \tag{5.6}$$

and hence, we find

$$r(u) = \frac{(4\pi m)^2}{\alpha^2} + \frac{4}{3\alpha^2 u^2}\,. \tag{5.7}$$

If we now multiply $r(u)$ by $(w(u) + w(-u))$, we find the equation

$$w^3(u) - r(u)w(u) = -w^3(-u) + r(-u)w(-u) \equiv s(u)\,. \tag{5.8}$$

The function $s(u)$ is clearly an odd function, and since $w(u)$ is regular for $\mathrm{Re}(u) < 0$, and $w(-u)$ is regular for $\mathrm{Re}(u) > 0$, $s(u)$ is analytic everywhere except at $u = 0$. Using eq. (5.6) and eq. (5.7), we find that $s(u)$ is determined and is given by

$$s(u) = \frac{16}{27\alpha^3 u^3} + 2\frac{(4\pi m)}{\alpha^2}\left(1 - \frac{2}{3\alpha}\right)\frac{1}{u}\,. \tag{5.9}$$

Hence, finding the resolvent has been reduced to solving a cubic equation. We do not actually need to do this, since to find the charges we only need to expand about $u = 0$. Given the generating function $t(u)$ in eq. (4.9), it is convenient to define the difference

$$\overline{w}(u) = w(u) - w(-u)\,. \tag{5.10}$$

Then it is simple to show using eq. (5.5) and eq. (5.8) that $\overline{w}(u)$ satisfies the cubic equation

$$\overline{w}^3(u) - r(u)\overline{w}(u) + s(u) = 0\,. \tag{5.11}$$

Solving eq. (5.11) as a series expansion, we find that the generator of the charges $t(u)$ is

$$\begin{aligned} t(u) &= -\frac{\alpha}{2}\left(\overline{w}(u) + W_r(u) - W_r(-u)\right) \\ &= \alpha(2\pi m)^2 u + \alpha(1-2\alpha)(2\pi m)^4 u^3 + \alpha(1 - 6\alpha + 7\alpha^2)(2\pi m)^6 u^5 + \cdots\,. \end{aligned} \tag{5.12}$$

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In particular, we note that

$$W'(0) = \frac{1}{2}\overline{w}'(0) = -(2\pi m)^2 , \tag{5.13}$$

and so

$$\gamma = \frac{\lambda m^2}{2L}\alpha = \frac{\lambda m^2 J'}{L^2} . \tag{5.14}$$

This is the same result obtained by Frolov and Tseytlin from the semiclassical string [12, 13]!

It is interesting to find the end points of the cut. At these points, $W(u)$ and hence $w(u)$ has a singularity. We can find these points by looking for the zeroes of the discriminant of eq. (5.11), which is

$$\begin{aligned}\Delta &= 4r^3(u) - 27s^2(u)\\ &= 64\frac{(2\pi m)^2}{\alpha^6 u^4}\left(4(1-\alpha) - (27\alpha^2 - 36\alpha + 8)(2\pi m)^2 u^2 + 4(2\pi m)^4 u^4\right) .\end{aligned} \tag{5.15}$$

This has zeroes at

$$u = \pm\frac{\sqrt{16 - 72\alpha + 54\alpha^2 \pm 2i\sqrt{\alpha(8-9\alpha)^3}}}{4(2\pi m)} , \tag{5.16}$$

with the four solutions corresponding to the endpoints of $\mathcal{C}_+$ and $\mathcal{C}_-$. Note that the system has an apparent critical point at $\alpha = 8/9$, where the end points of the cut hit the imaginary axis at

$$u = \pm\frac{i}{2\pi m\sqrt{3}} , \tag{5.17}$$

and the contours $\mathcal{C}_+$ and $\mathcal{C}_-$ touch each other. For this value of α we have $J' = 4J$. For $\alpha > 8/9$ the end points move along the imaginary axis, with two of them reaching $u = 0$ and the other two reaching $u = \pm i/4\pi m$ when $\alpha = 1$ and $J = 0$.

Strictly speaking, the Bethe states with $\alpha > 2/3$, which corresponds to $J' > J$, do not exist. This is because one of the Dynkin indices is negative in this region, and since the Bethe states are highest weights, the state must have zero norm. However, the anomalous dimension as well as all the higher charges are analytic across this point, so we believe that there will be another set of Bethe states with nonzero norm with these exact charges. This is sufficient since for an integrable system, the state is completely determined by the conserved charges. In the next section we will find Bethe states which are valid for at least for some of this region.

The conserved charges are also analytic across $\alpha = 8/9$. For this reason we believe the critical point is an artifact of the Bethe ansatz. In the next section we will show that these charges match with charges where the Bethe states exist, further demonstrating that the critical point is a fake. In anticipation of this, consider eq. (5.11) when $\alpha = 1$. In this case the solution for $\overline{w}(u)$ simplifies dramatically to

$$\begin{aligned}\overline{w}(u) &= -\frac{1}{3u} - \frac{\sqrt{1+(4\pi m)^2 u^2}}{u}\\ t(u) &= \frac{-1+\sqrt{1+(4\pi m)^2 u^2}}{2u} .\end{aligned} \tag{5.18}$$

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6. The gauge dual of the Frolov-Tseytlin string (II)

We now turn to case (iii) where $J' > J$. When $J = 0$ we only need one type of Bethe root to generate the Bethe state which means that this is a state for the Heisenberg XXX spin chain. The Bethe state dual to the Frolov-Tseylin $J = 0$ solution was constructed in [18] which we briefly review.

The starting point is the solution of the Bethe equations

$$u_{1,1} = 0\,, \qquad u_{1,2} = \frac{i}{2}\,, \qquad u_{1,3} = -\frac{i}{2}\,. \tag{6.1}$$

The equation for $u_{1,1}$ is identically satisfied provided L is even. The equations for the other two roots acquire the form $0 = 0$ or $\infty = \infty$. Appropriate infinitesimal shifts from $\pm i/2$ balance the singularities and renders the energy finite. In fact, a zero or a pole appears on the right hand side of the Bethe equations each time a pair of roots is separated by $\pm i$, which allows us to put extra roots on the imaginary axis. The left hand side of the Bethe equations will then be exponentially large or exponentially small in L. To compensate, the roots should be put exponentially close to $in/2$ with integer n's. This produces small denominators on the right hand side of the Bethe equations. If the number of the roots is macroscopic, the half-integer pattern breaks down at some critical $n \sim L$ [18] and the rest of the roots split along the imaginary axis by distances larger than $1/2$. If we parameterize the roots by $u_{1,k} = iq_{1,k}L$, and introduce the density

$$\sigma(q) = \frac{2}{L}\sum_k \delta(q - q_{1,k})\,, \tag{6.2}$$

it will consist of two parts, the condensate with $\sigma(q) = 4$ and two tails with $\sigma(q) < 4$. We write this as

$$\sigma(q) = \begin{cases} 4\,, & -s < q < s\,, \\ \tilde{\sigma}(q)\,, & s < q < t\,, \\ \tilde{\sigma}(-q)\,, & -t < q < -s\,, \\ 0\,, & |q| > t\,, \end{cases} \tag{6.3}$$

The Bethe equations unambiguously determine $\tilde{\sigma}(q)$ and the number of roots in the condensate, $4sL$. The parameter t is then fixed by the normalization condition

$$\int dq\,\sigma(q) = 2\alpha\,. \tag{6.4}$$

We have chosen to normalize the density differently than in the previous sections for later convenience, but αL is still the number of u_1 roots. Also the relationship between the number of the roots and the R-charges is different here. Explicit formulae for thr density in terms of elliptic integrals can be found in [18]. The results simplify considerably at $\alpha = 1/2$ where $t \to \infty$ and the density reduces to an algebraic function.

We can always add $u_{2,1} = 0$ to an arbitrary configuration of u_1 roots. This gives a solution of the Bethe equations with L replaced by $L + 1$ [16]. By numerically solving the Bethe equations for several configurations with a handful of u_2 roots, we observed that

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the u_2 roots cluster near zero on the imaginary axis. We therefore expect that in the thermodynamic limit the density of u_2 roots will differ from zero on an interval from $-v$ to $+v$ with $v < s$. As above, the density is defined by

$$\rho(q) = \frac{2}{L} \sum_k \delta(q - q_{2,k}) \,, \tag{6.5}$$

where $u_{2,k} = iq_{2,k}L$, and is normalized to

$$\int dq \, \rho(q) = 2\beta \,, \tag{6.6}$$

with βL equal to the number of u_2 roots.

Let us for the moment assume that the R-charges are arbitrary but the SYM dual is still holomorphic. In this case the number of roots of the Bethe state are related to the R-charges by $J_1 = (1-\alpha)L$, $J_2 = (\alpha - \beta)L$ and $J_3 = \beta L$. We will call the condition $J_1 = J_2$ "half-filling", and at half-filling we have that $2\alpha - \beta = 1$.

Taking the $L \to \infty$ limit of the Bethe equations, the roots outside of the condensate satisfy the equations

$$\begin{aligned} \frac{1}{q_{1,k}} &= \frac{2}{L} \sum_{l \neq k} \frac{1}{q_{1,k} - q_{1,l}} - \frac{1}{L} \sum_l \frac{1}{q_{1,k} - q_{2,l}} \,, \\ 0 &= \frac{2}{L} \sum_{l \neq k} \frac{1}{q_{2,k} - q_{2,l}} - \frac{1}{L} \sum_l \frac{1}{q_{2,k} - q_{1,l}} \,, \end{aligned} \tag{6.7}$$

or in terms of the densities,

$$2q \fint_s^t dq' \, \tilde{\sigma}(q') \, \frac{1}{q^2 - q'^2} = \frac{1}{q} + \frac{1}{2} \int_{-v}^{v} dq' \, \rho(q') \, \frac{1}{q - q'} - 4 \ln \frac{q - s}{q + s} \,, \qquad s < q < t \,, \tag{6.8}$$

$$\fint_{-v}^{v} dq' \, \rho(q') \, \frac{1}{q - q'} = q \int_s^t dq' \, \tilde{\sigma}(q') \, \frac{1}{q^2 - q'^2} + 2 \ln \frac{s + q}{s - q} \,, \qquad -v < q < v \,. \tag{6.9}$$

As in section 3, if we think of $q_{i,k}$ as coordinates of particles on a line subject to pairwise logarithmic interactions, these equations describe their equilibrium distribution. The logarithmic terms in the equations correspond to the interaction with the condensate in $\sigma(q)$. Roots of the same type repulse each other and roots of different types attract. In particular, q_1 roots create an effective potential for q_2 roots which has a minimum at zero and which confines the q_2 roots around the origin. This justifies our assumption about the functional form of the density of the second type of roots $\rho(q)$. The expression for the anomalous dimension depends only on the density of q_1 roots and so has the same form as for $\rho = 0$ [18], namely[1]

$$\gamma = \frac{\lambda}{8\pi^2 L} \left(\frac{4}{s} - \int_s^t dq \, \frac{\tilde{\sigma}(q)}{q^2} \right) . \tag{6.10}$$

There is a strong resemblance between the thermodynamic limit of the Bethe equations and saddle-point equations in large-N matrix models. In particular, the equation (6.9) is the same as the saddle-point equation in the hermitean one-matrix model [33], while (6.8)

[1] One has to carefully take into account the contribution of the condensate to derive this formula.

arises in large-N two-dimensional QCD on a sphere [34]. Our strategy will be to find $\tilde{\sigma}$ from the first equation, treating ρ as an external field and then to solve for ρ. The solution to (6.8) with a infinitesimal ρ was found in [18] by adapting the techniques of [34]. Since the equation is linear it is not hard to write down the general solution,

$$\begin{aligned}\tilde{\sigma}(q) &= \frac{1}{\pi}\sqrt{(q^2-s^2)(t^2-q^2)}\times \\ &\times\left[-\frac{1}{qst}+4\int_{-s}^{s}\frac{dx}{(q-x)\sqrt{(s^2-x^2)(t^2-x^2)}}-\frac{1}{2}\int_{-v}^{v}\frac{dx\,\rho(x)}{(q-x)\sqrt{(s^2-x^2)(t^2-x^2)}}\right].\end{aligned} \tag{6.11}$$

By plugging this expression into eq. (6.8), we get an additional constraint on s and t,

$$4\int_{-s}^{s}\frac{dx}{\sqrt{(s^2-x^2)(t^2-x^2)}}=\frac{1}{st}+\frac{1}{2}\int_{-v}^{v}\frac{dx\,\rho(x)}{\sqrt{(s^2-x^2)(t^2-x^2)}}\,. \tag{6.12}$$

Another constraint is derived from the normalization conditions eq. (6.4) and eq. (6.6),

$$4\int_{-s}^{s}\frac{dx\,x^2}{\sqrt{(s^2-x^2)(t^2-x^2)}}-\frac{1}{2}\int_{-v}^{v}\frac{dx\,\rho(x)x^2}{\sqrt{(s^2-x^2)(t^2-x^2)}}=1-2\alpha+\beta\,. \tag{6.13}$$

Since $\rho(x)<4$, the left hand side of eq. (6.13) is manifestly positive. Half-filling $1-2\alpha+\beta=0$ therefore corresponds to a critical point at which t goes to infinity. The density then simplifies to

$$\tilde{\sigma}(q)=\left(4-\frac{1}{\pi s}\sqrt{1-\frac{s^2}{q^2}}-\frac{1}{2\pi}\int_{-v}^{v}dx\,\frac{\rho(x)}{q-x}\sqrt{\frac{q^2-s^2}{s^2-x^2}}\right) \tag{6.14}$$

and the consistency condition eq. (6.12) becomes

$$4\pi=\frac{1}{s}+\frac{1}{2}\int_{-v}^{v}\frac{dx\,\rho(x)}{\sqrt{s^2-x^2}}\,. \tag{6.15}$$

This condition ensures that the density decreases at infinity. We can also compute the anomalous dimension which is given by

$$\gamma=\frac{\lambda}{32\pi^2L}\left[\frac{1}{s^2}+\int_{-v}^{v}dq\,\rho(q)\,\frac{1}{q^2}\left(\frac{s}{\sqrt{s^2-q^2}}-1\right)\right]. \tag{6.16}$$

The next step is to substitute $\tilde{\sigma}$ into (6.9) and to solve for $\rho(q)$. We then find

$$\fint_{-v}^{v}\frac{dx\,\rho(x)}{q-x}\left(3+\sqrt{\frac{s^2-q^2}{s^2-x^2}}\right)=\frac{2}{q}\left(1-\sqrt{1-\frac{q^2}{s^2}}\right). \tag{6.17}$$

This equation has a simpler analytic structure than it may seem because the square roots can be eliminated by a simple change of variables:

$$q=\frac{2s\eta}{1+\eta^2}\,,\qquad x=\frac{2s\xi}{1+\xi^2}\,,\qquad dx\rho(x)=d\xi\rho(\xi)\,. \tag{6.18}$$

In the new variables the integral equation is

$$\fint d\xi\,\rho(\xi)\,\frac{1+\xi^2}{1-\xi^2}\left(2\,\frac{1+\eta\xi}{\eta-\xi}+\frac{\eta+\xi}{1-\eta\xi}\right)=2\eta\,, \tag{6.19}$$

where it contains only rational coefficients.

The consistency condition (6.15) then reads

$$\int d\xi\,\rho(\xi)\,\frac{1+\xi^2}{1-\xi^2}=8\pi s-2\,, \tag{6.20}$$

and the anomalous dimension is

$$\gamma=\frac{\lambda}{32\pi^2 Ls^2}\left[1+\frac{1}{2}\int d\xi\,\rho(\xi)\,\frac{(1+\xi^2)^2}{1-\xi^2}\right]. \tag{6.21}$$

After the further change of variables, $\xi=(1-p)/(1+p)$, eq. (6.19) reduces to the $O(n)$ matrix model form in eq. (3.11), but now with $n=1$. However, we found it more practical to do the calculation in the original variables while repeating the steps in [29]. To this end, let us define the resolvent

$$F(z)=\int d\xi\,\rho(\xi)\,\frac{1+\xi^2}{1-\xi^2}\,\frac{1+z\xi}{z-\xi}\,. \tag{6.22}$$

$F(z)$ is an odd function whose only singularities are a pole at infinity given by

$$F(z)=\frac{p}{z}+\cdots\,,\qquad (z\to\infty)\,, \tag{6.23}$$

and a branch cut from $-\nu$ to ν, where ν is related to v by

$$v=\frac{2s\nu}{1+\nu^2}\,. \tag{6.24}$$

The branch points lie inside the unit circle because v is smaller than s. When the density of u_2 roots approaches the end-point of the condensate, ν goes to one. The residue at infinity is given by

$$p=\int d\xi\,\rho(\xi)\,\frac{(1+\xi^2)^2}{1-\xi^2}\,,$$

and hence

$$\gamma=\frac{\lambda}{32\pi^2 Ls^2}\left(1+\frac{p}{2}\right). \tag{6.25}$$

The consistency condition (6.20) and the normalization eq. (6.6) can be easily expressed in terms of $F(z)$

$$F(i)=-i(8\pi s-2)\,, \tag{6.26}$$

$$F'(i)=2\beta\,. \tag{6.27}$$

The resolvent satisfies a functional equation which can be derived by multiplying both sides of (6.19) by

$$\rho(\eta)\,\frac{1+\eta^2}{1-\eta^2}\left(\frac{1}{z-\eta}-\frac{1}{z-1/\eta}\right)$$

and integrating over η. A long but straightforward calculation yields

$$F^2(z)+F^2\left(\frac{1}{z}\right)+F(z)F\left(\frac{1}{z}\right)-2zF(z)-\left(\frac{2}{z}\right)F\left(\frac{1}{z}\right)+64\pi^2s^2-4=0\,. \tag{6.28}$$

In order to get rid of the linear terms, we expand $F(z)$ as

$$F(z) = \frac{4z}{3} - \frac{2}{3z} + w(z)\,. \tag{6.29}$$

$w(z)$ satisfies the purely quadratic equation

$$w^2(z) + w^2\left(\frac{1}{z}\right) + w(z)w\left(\frac{1}{z}\right) = R(z)\,, \tag{6.30}$$

where

$$R(z) = R\left(\frac{1}{z}\right) = \frac{4}{3}\left(z + \frac{1}{z}\right)^2 - 64\pi^2 s^2\,. \tag{6.31}$$

Multiplying (6.30) by $w(z) - w(1/z)$, we again find a cubic equation

$$w^3(z) - R(z)w(z) = w^3\left(\frac{1}{z}\right) - R\left(\frac{1}{z}\right) w\left(\frac{1}{z}\right) \equiv S(z)\,. \tag{6.32}$$

The manifestly odd function $S(z)$ is symmetric under $z \to 1/z$, has poles at zero and at infinity and potentially has branch points at $\pm\nu$, $\pm 1/\nu$. But the left hand side of (6.32) is analytic at $\pm 1/\nu$ and the middle is analytic at $\pm\nu$. Consequently, $S(z)$ has no branch points, its only singularities are poles at zero and at infinity, therefore it is an odd polynomial in $z + 1/z$ of at most third degree. Taking into account the definition of $w(z)$, eq. (6.29), and the boundary condition (6.23), we get

$$S(z) = -\frac{16}{27}\left(z + \frac{1}{z}\right)^3 + \frac{4}{3}(6 + 3p - 64\pi^2 s^2)\left(z + \frac{1}{z}\right). \tag{6.33}$$

Solving the cubic equation (6.32), we can find $w(z)$ and therefore $F(z)$. However, we do not need the explicit form of the resolvent to compute the anomalous dimension. We only need to know the constant p and that is determined by the constraints (6.26), (6.27). Putting $z = i$ in (6.32) and taking into account (6.26) gives an identity and does not lead to any relation between the parameters. But if we first differentiate in z and then put $z = i$, we find the non-trivial equation for p in terms of s and β,

$$p = 32\pi^2 s^2 (1 - \beta) - 2\,. \tag{6.34}$$

Therefore, the anomalous dimension again has the amazingly simple form

$$\gamma = \frac{\lambda(1 - \beta)}{2L} = \frac{\lambda J'}{L^2}\,, \tag{6.35}$$

which agrees with the string-theory prediction of Frolov and Tseytlin [12], and is consistent with the solution in the previous section.

Let us now see how the number of roots in the condensate depends on the total number of u_2 roots, i.e. the relation of s to β. There are no other constraints or boundary conditions imposed on the resolvent $F(z)$ than those that we have already used to find the anomalous dimension, so the dependence of s on β must be determined by the analytic structure of $F(z)$. Like the previous section, we examine the discriminant of the cubic equation (6.32),

$$\Delta(z) = 4R^3(z) - 27\, S^2(z)\,. \tag{6.36}$$

The solution of the cubic equation and therefore the resolvent $F(z)$ depends on z through $\sqrt{\Delta(z)}$. Single zeros of the discriminant (but not double zeros!) are branch points of the resolvent. Using the explicit expressions for $R(z)$ and $S(z)$ we find

$$\Delta = -4096\pi^2 s^2 \beta \left[\left(z+\frac{1}{z}\right)^2 - 2\pi^2 s^2 \frac{\chi_+}{\beta}\right]\left[\left(z+\frac{1}{z}\right)^2 - 2\pi^2 s^2 \frac{\chi_-}{\beta}\right], \tag{6.37}$$

where

$$\chi_\pm = 1 + 18\beta - 27\beta^2 \pm (1-9\beta)\sqrt{(1-9\beta)(1-\beta)}\,. \tag{6.38}$$

The discriminant Δ, as a function of $z+1/z$, has four zeros. Consequently, the resolvent will have four branch points instead of two, unless $2\pi^2 s^2 \chi_-/\beta = 4$ in which case the discriminant has a double zero at $z = \pm 1$. Insisting on only two branch points we find

$$s = \frac{\sqrt{2 + 36\beta - 54\beta^2 + 2\sqrt{(1-\beta)(1-9\beta)^3}}}{8\pi}\,. \tag{6.39}$$

Examining eq. (6.39) we see that s grows with β because u_1 roots in the tail of the distribution are attracted toward the u_2 roots at the origin. The parameter s changes from $1/(4\pi)$ at $\beta = 0$ to $1/(2\pi\sqrt{3})$ at $\beta = 1/9$ and becomes complex for larger β. But s is real by definition, hence $\beta = 1/9$ is a critical point for the Bethe equations. In terms of R-charges, $\beta = 1/9$ corresponds to $J' = 4J$, the point we found in section 5.

Let us now look at the branch points of the resolvent:

$$v = \frac{\sqrt{2 + 36\beta - 54\beta^2 - 2\sqrt{(1-\beta)(1-9\beta)^3}}}{8\pi}\,. \tag{6.40}$$

As $\beta \to 1/9$, $v \to s$ and the density ρ of u_2 roots collides with the tail of the distribution of u_1 roots, $\tilde{\sigma}$. This type of critical behavior corresponds to the Ising phase transition in the $O(1)$ matrix model [29]. But in the case of the string, we believe that it is only a signal that this particular configuration of Bethe roots can no longer describe the string state and that the physical system happily continues through this point. Indeed, if we compare eq. (6.39) and eq. (6.40) with the end points of the cuts in eq. (5.16), and recall that α in eq. (5.16) is $2J'/L$ and is related to β by $\alpha = 1 - \beta$, we see that the end points of $\mathcal{C}_+$ match onto s and $-v$ and the end points of $\mathcal{C}_-$ match onto $-s$ and v. This strongly suggests that the analytic continuation of the $J' < J$ sector is the sector described in this section, or at least if $J' > 4J$.

As a further check on the spurious nature of the critical point let us consider the higher conserved charges when $J = 0$, which is in the $J' > 4J$ region. The contribution of the condensate to $t(u)$ in eq. (4.8), after rescaling, is

$$t_c(u) = i \prod_{n=-2sL}^{2sL} \log\left(\frac{uL - ni/2 + i/2}{uL - ni/2 - i/2}\right) = 2i \log\frac{u+is}{u-is}\,. \tag{6.41}$$

Using the normalization condition in eq. (6.4), one finds

$$t(u) = 2i\log\frac{u+is}{u-is} - \frac{1}{2}\left(\int_{-\infty}^{-s} dq \frac{\tilde{\sigma}(q)}{u-iq} + \int_s^\infty dq \frac{\tilde{\sigma}(q)}{u-iq}\right). \tag{6.42}$$

In [18] it was shown that

$$\tilde{\sigma}(q) = 4\left(1 - \sqrt{1 + \frac{s^2}{q^2}}\right) \tag{6.43}$$

when $J = 0$. Inserting eq. (6.43) into eq. (6.42) and deforming the contour, leads to

$$t(u) = 2\pi\left(-\frac{s}{u} + \sqrt{1 + \frac{s^2}{u^2}}\right) . \tag{6.44}$$

Since $s = 1/4\pi$ when $J = 0$, we see that this is the $m = 1$ result in eq. (5.18). Hence, all conserved charges for the $J' < J$ Bethe states continue through onto the $J = 0$ Bethe state.

7. Discussion

Combining the results of this paper with [18], we can summarize the SYM duals of different semiclassical string motion as follows: The folded string is dual to an $O(-2)$ model, the circular string is dual to an $O(-1)$ or $O(+1)$ model and the pulsating string is dual to an $O(0)$ model. A natural question to ask is whether other $O(n)$ models are related to yet other types of semiclassical string motion. One might also wonder if there is a deeper connection between the strings in $AdS_5 \times S^5$ and the matrix models. For example, the $O(n)$ models are known to be critical only within the range $-2 < n < 2$, hence -2 is a limiting value. The same is true of the folded string, in the sense that this is a limit of an ellipsoidal string that smoothly interpolates into a circular string.

Another interesting question concerns the stability of the semiclassical strings. In [12] it was shown that the single wound circular string develops an unstable mode at $J' \geq 3J/2$. Presumably this can be checked on the dual side, where one can look for the spinless modes by shifting roots around. This was accomplished for the $J = 0$ case in [18] and we expect the same techniques to work when $J \neq 0$.

It would also be interesting to extend these results to $\alpha < 0$. This was done in [21] to the results of [18]. The authors showed that the resulting integral equation was the thermodynamic limit of a particular sector of the $\mathrm{SU}(2,2|4)$ Bethe equations in [20]. They then went on to show that anomalous dimension matched the predicted anomalous dimension for a folded string spinning in AdS_5 and with angular momentum in S_5 [8]. We expect a similar phenomenon to occur here.

Another worthwhile goal is to better understand the higher charges from the point of view of the semiclassical string. It is not immediately clear what these conserved charges would mean. The semiclassical string contains information about all orders in λ, but the integrability of the dilatation operator has been established only at one-loop, although there are hints that integrability can be taken further in SYM [35, 36], as well as cautions [37]. Nevertheless, it has been recently pointed out that the $AdS_5 \times S_5$ sigma model has an infinite tower of non-local conserved charges [38, 39]. This was shown to also hold in the Berkovits description [40] and was further analyzed in the plane-wave limit [41]. In [42], some progress was made toward relating the integrability of the one-loop dilatation operator and the non-local symmetries in the sigma model. Hopefully, the elegant yet simple equation in eq. (4.9) relating the generator of conserved charges to the resolvent of the Bethe roots will provide further clues.

Note added: after completion of this work we learned about work of Arutyunov and Staudacher [43] where they show that higher charges coincide on both sides of the AdS/CFT correspondence for the solutions in [12, 14, 18].

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